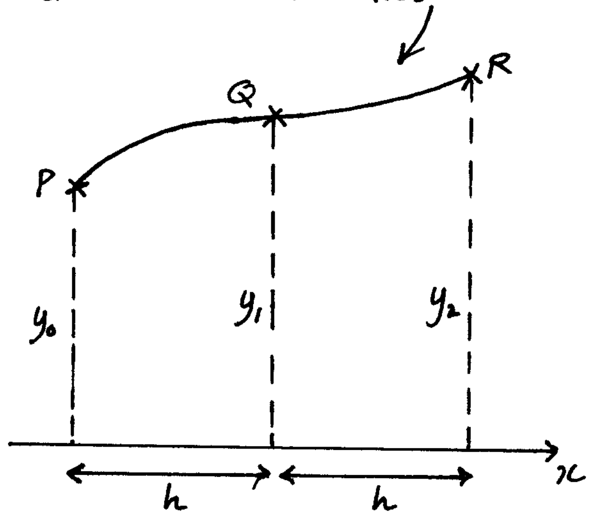
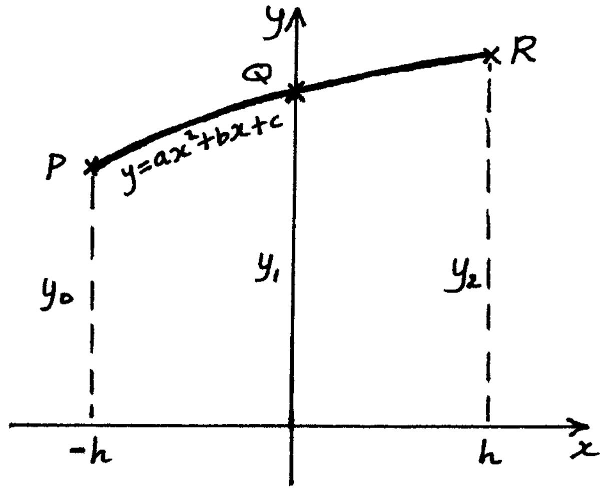
NUMERICAL TECHNIQUES OF INTEGRATION – SIMPSON'S RULE

Suppose we want to estimate the area between a curve and the x-axis





Divide the area into 2 strips.

Move the points P, Q, R to the above positions.

The parabola $y = ax^2 + bx + c$ is drawn through the points P, Q, R.

area under parabola between P and R

$$= \int_{-h}^{h} y \, dx$$

$$= \int_{-h}^{h} ax^2 + bx + c \quad dx$$

$$= \left[\tfrac{1}{3}ax^3 + \tfrac{1}{2}bx^2 + cx\right]_{-h}^{h}$$

$$= \left(\tfrac{1}{3}ah^3 + \tfrac{1}{2}bh^2 + ch\right) - \left(-\tfrac{1}{3}ah^3 + \tfrac{1}{2}bh^2 - ch\right)$$

$$= \tfrac{2}{3}ah^3 + 2ch$$

$$= \tfrac{1}{3}h\left(2ah^2 + 6c\right)$$

Consider equation of parabola $y = ax^2 + bx + c$:

subst. $x = -h$, $y = y_0$: $\quad ah^2 - bh + c = y_0 \quad \ldots (1)$

subst. $x = 0$, $y = y_1$: $\quad c = y_1 \quad \ldots (2)$

Subst. $x = h$, $y = y_2$: $\quad ah^2 + bh + c = y_2 \quad \ldots (3)$

$(1)+(3)$: $\quad 2ah^2 + 2c = y_0 + y_2 \quad \ldots (4)$

Subst. (2) in (4): $\quad 2ah^2 = y_0 - 2y_1 + y_2 \quad \ldots (5)$

Subst. (2) and (5) in result for area under parabola:

area under parabola

$$= \tfrac{1}{3}h\,(y_0 - 2y_1 + y_2 + 6y_1)$$

$$= \tfrac{1}{3}h\,(y_0 + 4y_1 + y_2)$$

$\therefore$ area under original curve

$$\approx \tfrac{1}{3}h\,(y_0 + 4y_1 + y_2)$$

Now divide the area into 10 strips of width h:

total area under the 5 parabolas

$$\begin{aligned} = \; & \tfrac{1}{3}h\,(y_0 + 4y_1 + y_2) \\ & + \tfrac{1}{3}h\,(y_2 + 4y_3 + y_4) \\ & + \tfrac{1}{3}h\,(y_4 + 4y_5 + y_6) \\ & + \tfrac{1}{3}h\,(y_6 + 4y_7 + y_8) \\ & + \tfrac{1}{3}h\,(y_8 + 4y_9 + y_{10}) \end{aligned}$$

$$= \tfrac{1}{3}h\,(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + 2y_6 + 4y_7 + 2y_8 + 4y_9 + y_{10})$$

In general, if the area is divided into an <u>even</u> number n of strips of width h:

$$\text{area} \approx \tfrac{1}{3}h\,(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \dots + 4y_{n-1} + y_n)$$

Compare with the trapezoidal rule:

$$\text{area} \approx \tfrac{1}{2}h\,(y_0 + 2y_1 + 2y_2 + 2y_3 + 2y_4 + \dots + 2y_{n-1} + y_n)$$

NB. Simpson's Rule must give the exact result for the area under a parabola. It can be shown that it gives the exact result under a cubic curve. As part of a proof of this, it can be shown that $\int_a^{a+2h} x^3\,dx = \tfrac{1}{3}h\,(y_0 + 4y_1 + y_2)$.